

Blowup of Regular Solutions for the Relativistic Euler-Poisson Equations

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Abstract

In this paper, we study the blowup phenomena for the regular solutions of the isentropic relativistic Euler-Poisson equations with a vacuum state in spherical symmetry. Using a general family of testing functions, we obtain new blowup conditions for the relativistic Euler-Poisson equations. We also show that the proposed blowup conditions are valid regardless of the speed requirement, which was one of the key constraints stated in "Y. Geng, *Singularity Formation for Relativistic Euler and Euler-Poisson Equations with Repulsive Force*, Commun. Pure Appl. Anal., **14** (2015), 549–564."

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1 Introduction

The isentropic relativistic Euler-Poisson equations [1] are expressed as follows:

$$\left\{ \begin{array}{l} \partial_t \left(\frac{n}{\sqrt{1-|v|^2/c^2}} \right) + \nabla \cdot \left(\frac{nv}{\sqrt{1-|v|^2/c^2}} \right) = 0, \\ \partial_t \left(\frac{p/c^2 + \rho}{1-|v|^2/c^2} v \right) + \nabla \cdot \left(\frac{p/c^2 + \rho}{1-|v|^2/c^2} v \otimes v \right) + \nabla p = \frac{4\pi n \nabla \phi}{\sqrt{1-|v|^2/c^2}}, \\ \Delta \phi = \frac{4\pi n}{\sqrt{1-|v|^2/c^2}}, \end{array} \right. \quad (1)$$

where n is defined by

$$\rho = n (1 + e/c^2), \quad (2)$$

which satisfies

$$\frac{dn}{nc^2} = \frac{d\rho}{p + \rho c^2}. \quad (3)$$

The unknowns and constants in the above equation are defined as follows:

$\rho : [0, \infty) \times \mathbf{R}^3 \rightarrow [0, \infty)$ and $n : [0, \infty) \times \mathbf{R}^3 \rightarrow [0, \infty)$ denote the proper mass-energy density and the charge density, respectively; c is the speed of light; $v : [0, \infty) \times \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is the velocity of an electro-fluid; $-\phi : [0, \infty) \times \mathbf{R}^3 \rightarrow \mathbf{R}$ is the electrostatic potential in the inertial frame; and $p = p(\rho)$ is the pressure function of the electro-fluid in a proper frame. The equation of state p follows the γ -law:

$$p = \rho^\gamma, \quad (4)$$

where $\gamma > 1$ is the adiabatic index. Lastly, the constant $e \geq 0$ in (2) is the specific internal energy.

Relativistic electrodynamics includes the study of the interaction between relativistic charged particles and electromagnetic fields when the particles are moving at a speed comparable to the speed of light in a vacuum. At such a high speed, the motion of the charged particles no longer obeys the Newtonian equations, so relativistic equations of particles must be applied. Under a field with a much stronger electric than magnetic effect, such as those in supernova explosions, gravitational collapse, and the formation and expansion of black holes and neutron stars, the

motion of an isentropic relativistic electro-fluid can be described by the Euler-Poisson equations (2) when the charged particles are moving very fast.

To understand the mathematical nature of relativistic fluid dynamics, we first review a related and previously developed relativistic model, namely, the relativistic Euler equations. Makino and Ukai [9, 10] and Lefloch and Ukai [5] established the local existence of classical solutions to the relativistic system using the theory of a quasi-linear symmetric hyperbolic system. More precisely, the critical part of Makino and Ukai's proof was based on the existence of a strictly convex entropy for the non-vacuum case; the critical part of Lefloch and Ukai's proof relied on the generalized Riemann invariants and normalized velocity for the vacuum case. Geng and Li [2] extended these results to the isentropic system. For the non-isentropic system, Guo [4] proved the blowup result for smooth solutions using the averaged quantities method developed by Sideris [13, 14]. Moreover, Pan and Smoller [11] applied the classical energy method to show the singularity formation of smooth solutions.

Due to the complexity of the structures of system (1), research on multi-dimensional relativistic Euler-Poisson equations is still at an early stage. In 2013, Mai, Li and Zhang [7] gave the first well-posed result for the steady-state relativistic Euler-Poisson equations with relaxation. For system (1) in the one dimensional case, Geng and Wang [3] obtained the global existence of a smooth solution with some monotonic conditions on the initial data. The importance of system (1) is that the non-relativistic Euler-Poisson equations are the Newtonian limit of system (1). Readers may refer to [12, 18, 19, 15] for the blowup results of the non-relativistic Euler-Poisson equations.

In this paper, we consider the spherical symmetric solutions, namely,

$$n = n(t, r), \quad \rho = \rho(t, r), \quad v = \frac{x}{r}v(t, r), \quad (5)$$

where $r = |x|$ is the radius of the spatial variables $x \in \mathbf{R}^3$.

Then, system (1) is transformed into

$$\left\{ \begin{array}{l} \partial_t \left(\frac{n}{\sqrt{1-v^2/c^2}} \right) + \partial_r \left(\frac{n}{\sqrt{1-v^2/c^2}} v \right) + \frac{2}{r} \frac{n}{\sqrt{1-v^2/c^2}} v = 0, \\ \partial_t \left(\frac{p/c^2 + \rho}{1-v^2/c^2} v \right) + \partial_r \left(\frac{p/c^2 + \rho}{1-v^2/c^2} v^2 \right) + \partial_r p + \frac{2}{r} \frac{p/c^2 + \rho}{1-v^2/c^2} v^2 = \frac{4\pi n \phi_r}{\sqrt{1-v^2/c^2}}, \\ \phi_{rr} = -\frac{8\pi}{r^2} \int_0^r \frac{n}{\sqrt{1-v^2/c^2}} s^2 ds + \frac{4\pi n}{\sqrt{1-v^2/c^2}}. \end{array} \right. \quad (6)$$

Note that we have

$$\phi_r = \frac{4\pi}{r^2} \int_0^r \frac{n}{\sqrt{1-v^2/c^2}} s^2 ds \quad (7)$$

as equation (6)₃ can be solved using Green's function.

In [18], Yuen obtained a blowup result for the compressible Euler and Euler-Poisson equations with repulsive forces using the integration method. Recently, Geng [1] used the integration method described in [18] to obtain a blowup result for the regular solutions of relativistic Euler and Euler-Poisson equations. Note that in [16, 17], the authors generalized the testing function in the integration method to any strictly increasing function. By combining the method in [16, 17] with the results of Geng [1], we obtain the following theorem, which is our main contribution. Moreover, we remove the condition that $|v| \geq c/2$ in [1].

Theorem 1 *For any strictly increasing C^1 function $f(r)$ vanishing at $r = 0$, the regular solutions of system (6) with initial data (12) and*

$$p'(\rho) < ac^2, \quad \text{for some } a \in (0, 1) \quad (8)$$

blow up on or before the finite time

$$T = \frac{2 \left(\int_0^R f(r) v_0 dr \right) \left(\int_0^R \frac{f^2(r)}{f'(r)} dr \right)}{\left(\int_0^R f(r) v_0 dr \right)^2 - 2C \left(\int_0^R \frac{f^2(r)}{f'(r)} dr \right) \left(\int_0^R f(r) dr \right)} > 0, \quad (9)$$

if

$$\int_0^R f(r) v_0 dr > \sqrt{2C \left(\int_0^R \frac{f^2(r)}{f'(r)} dr \right) \left(\int_0^R f(r) dr \right)}, \quad (10)$$

where

$$C = \frac{c^2(\gamma + a - \gamma a + 9)}{2(\gamma - 1)(1 - a)^2} > 0. \quad (11)$$

2 Integration Method

We consider the Cauchy problem of system (6) with the following initial data:

$$\begin{cases} \rho(0, r) = \rho_0(r) \geq 0, \\ v(0, r) = v_0(r) < c, \\ \text{supp}(\rho_0, v_0) \subseteq \{r : r \leq R\}, \quad \text{for some } R > 0. \end{cases} \quad (12)$$

Note that for system (6) to be well-defined, we must have

$$|v| < c, \quad (13)$$

which is guaranteed by (12)₂.

First, we show that the sign of ρ is determined by its initial value.

Lemma 2 *If n_0 is positive, then n is always positive, where n_0 is defined by*

$$\rho_0 = n_0 (1 + e/c^2). \quad (14)$$

Proof. Let $r(t; r_0)$ be a characteristic curve starting at r_0 . That is, $r(t; r_0)$ satisfies the following ordinary differential equation:

$$\begin{cases} \frac{d}{dt} r(t; r_0) = v(t, r(t; r_0)), \\ r(0; r_0) = r_0. \end{cases} \quad (15)$$

Then, for any C^1 function $F(t, r)$, by the chain rule, we have

$$\frac{d}{dt} F(t, r(t; r_0)) = F_t(t, r(t; r_0)) + v(t, r(t; r_0)) F_r(t, r(t; r_0)). \quad (16)$$

Take

$$F = \frac{n}{\sqrt{1 - v^2/c^2}}, \quad (17)$$

then (6)₁ becomes

$$\frac{d}{dt} F + (v_r + 2v/r) F = 0, \quad (18)$$

where we omit the substitution $r = r(t; r_0)$. The above ordinary differential equation can be solved using an integral factor and the solution is

$$F(t, r(t; r_0)) = F(0, r_0) \exp \left(- \int_0^t (v_r + 2v/r)(s, r(s; r_0)) ds \right). \quad (19)$$

Thus,

$$n_0 > 0 \Rightarrow F(0, r_0) > 0 \Rightarrow F(t, r(t; r_0)) > 0 \Rightarrow n(t, r(t; r_0)) > 0. \quad (20)$$

The proof is complete. ■

Remark 3 By (2), ρ and n have the same sign. Thus, it follows from the initial data (12) that ρ is always non-negative.

Lemma 4 For any C^1 non-vacuum ($\rho \neq 0$) solutions of system (6) with the condition

$$p'(\rho) < c^2, \quad (21)$$

we have the following relation:

$$v_t + \frac{1 - p'/c^2}{1 - p'v^2/c^4}vv_r + \frac{(1 - v^2/c^2)^2 p'}{q(1 - p'v^2/c^4)}\rho_r = \frac{4\pi n(\sqrt{1 - v^2/c^2})^3}{q(1 - p'v^2/c^4)}\phi_r + \frac{2(1 - v^2/c^2)v^2 p'/c^2}{r(1 - p'v^2/c^4)}, \quad (22)$$

where

$$q := p/c^2 + \rho \quad (23)$$

and

$$p' := p'(\rho), \quad \text{is the derivative of } p \text{ with respect to } \rho. \quad (24)$$

Proof. From (3), we have

$$\frac{dn}{n} = \frac{d\rho}{q}. \quad (25)$$

From (6)₁, we have

$$\begin{aligned} \frac{n}{q\sqrt{1 - v^2/c^2}}\rho_t + \frac{n}{c^2 \left(\sqrt{1 - v^2/c^2}\right)^3}vv_t + \frac{n}{q\sqrt{1 - v^2/c^2}}v\rho_r + \frac{n}{\left(\sqrt{1 - v^2/c^2}\right)^3}v_r \\ + \frac{2n}{r\sqrt{1 - v^2/c^2}}v = 0 \end{aligned} \quad (26)$$

or

$$\rho_t + v\rho_r = -\frac{q}{c^2(1 - v^2/c^2)}vv_t - \frac{q}{1 - v^2/c^2}v_r - \frac{2qv}{r}. \quad (\text{For } n \neq 0 \text{ and } q \neq 0) \quad (27)$$

On the other hand, from (6)₂, we have

$$\frac{1 + p'/c^2}{1 - v^2/c^2}(\rho_t + v\rho_r)v + \frac{q(1 + v^2/c^2)}{(1 - v^2/c^2)^2}v_t + \frac{2q}{(1 - v^2/c^2)^2}vv_r + p'\rho_r + \frac{2q}{r(1 - v^2/c^2)}v^2 = \frac{4\pi n}{\sqrt{1 - v^2/c^2}}\phi_r. \quad (28)$$

Substituting (27) into (28), we have

$$\frac{(1 - p'v^2/c^4)q}{(1 - v^2/c^2)^2}v_t + \frac{(1 - p'/c^2)q}{(1 - v^2/c^2)^2}vv_r - \frac{2qp'/c^2}{r(1 - v^2/c^2)}v^2 + p'\rho_r = \frac{4\pi n}{\sqrt{1 - v^2/c^2}}\phi_r \quad (29)$$

or (for $q \neq 0$),

$$v_t + \frac{1 - p'/c^2}{1 - p'v^2/c^4}vv_r + \frac{(1 - v^2/c^2)^2p'}{q(1 - p'v^2/c^4)}\rho_r = \frac{4\pi n(\sqrt{1 - v^2/c^2})^3}{q(1 - p'v^2/c^4)}\phi_r + \frac{2(1 - v^2/c^2)v^2p'/c^2}{r(1 - p'v^2/c^4)}. \quad (30)$$

The proof is complete. ■

Following Makino, Ukai and Kawashima [8], Geng in [1] introduced a corresponding notion of regular solutions for the relativistic Euler-Poisson system, defined as follows.

Definition 5 *A solution (ρ, v) of system (6) is regular if the following conditions are satisfied:*

(1)

$$\left(p^{\frac{\gamma-1}{2\gamma}}, v\right) \in C^1. \quad (31)$$

(2)

$$|(v^2)_r| \leq c^2. \quad (32)$$

(3)

$$|(p')_r| \leq c^2. \quad (33)$$

Remark 6 *Condition (1) in the above definition implies that a regular solution is C^1 .*

In [1], Geng showed the non-expanding support of the regular solutions of system (6). For the sake of completeness, we include a proof here.

Proposition 7 *The supports for the regular solutions of system (6) with initial conditions (12) do not expand. That is to say,*

$$\text{supp}(\rho, v) \subseteq \{r : r \leq R\} \quad \text{for all } t.$$

Proof. As in the proof of Lemma 4, let $r(t; r_0)$ be a characteristic curve starting at r_0 . Then, when $r_0 \geq R$, $\rho_0(r_0) = n_0(r_0) = 0$. From (19),

$$n(t, r(t; r_0)) = \rho(t, r(t; r_0)) = 0. \quad (34)$$

On the other hand, note that for $\rho \neq 0$,

$$\frac{p'}{q}\rho_r = \frac{\gamma\rho^{\gamma-2}\rho_r}{\rho^{\gamma-1}/c^2 + 1} = \frac{2\gamma/(\gamma-1)}{w^2/c^2 + 1}ww_r \quad (35)$$

and

$$\frac{n}{q} = \frac{1}{(1 + e/c^2)(\rho^{\gamma-1}/c^2 + 1)}, \quad (36)$$

where

$$w := p^{\frac{\gamma-1}{2\gamma}} \quad (37)$$

is C^1 . Thus, by continuity, when $\rho = 0$, $w = 0$, and hence

$$\frac{p'}{q}\rho_r = 0. \quad (38)$$

Similarly, when $\rho = 0$, by continuity we have

$$\frac{n}{q} = \frac{1}{1 + e/c^2}. \quad (39)$$

Thus, when $r_0 \geq R$, after substitution, (22) becomes

$$v_t + vv_r = 0. \quad (\text{Note that } p' = 0 \text{ when } \rho = 0.) \quad (40)$$

From (16) ($F = v$), we have

$$\frac{d}{dt}v(t, r(t; r_0)) = 0 \quad (41)$$

or

$$v(t, r(t; r_0)) = v_0(r_0) = 0. \quad (42)$$

From (15), we have

$$\frac{d}{dt}r(t; r_0) = v(t, r(t; r_0)) = 0, \quad \text{when } r_0 \geq R. \quad (43)$$

Thus,

$$r(t; r_0) = r(0; r_0) = r_0, \quad \text{when } r_0 \geq R. \quad (44)$$

Therefore, we have

$$\rho(t, r_0) = \rho(t, r(t; r_0)) = 0 \quad (45)$$

and

$$v(t, r_0) = v(t, r(t; r_0)) = 0 \quad (46)$$

when $r_0 \geq R$.

As $r_0 \geq R$ is arbitrary, the proof is complete. ■

Now, we are ready to present the proof of the theorem.

Proof of Theorem 1. By Lemma 4 and (7), we have

$$v_t + \frac{1 - p'/c^2}{1 - p'v^2/c^4}vv_r + \frac{(1 - v^2/c^2)^2p'}{q(1 - p'v^2/c^4)}\rho_r = \frac{4\pi n(\sqrt{1 - v^2/c^2})^3}{q(1 - p'v^2/c^4)}\phi_r + \frac{2(1 - v^2/c^2)v^2p'/c^2}{r(1 - p'v^2/c^4)} \geq 0. \quad (47)$$

Thus,

$$v_t + vv_r - \frac{p'/c^2(1 - v^2/c^2)}{1 - p'v^2/c^4}vv_r + \frac{(1 - v^2/c^2)^2p'}{q(1 - p'v^2/c^4)}\rho_r \geq 0. \quad (48)$$

Now multiply $f(r)$ and take integration over $[0, R]$ with respect to r on both sides of (48) to obtain

$$\int_0^R f(r)v_t dr + \int_0^R f(r)vv_r dr + \int_0^R f(r)\frac{p'/c^2(1 - v^2/c^2)}{1 - p'v^2/c^4}(-vv_r)dr + \int_0^R f(r)\frac{(1 - v^2/c^2)^2p'}{q(1 - p'v^2/c^4)}\rho_r dr \geq 0, \quad (49)$$

where R is given by the initial data (12).

First, the integrand of the third term in expression (49) is less than or equal to

$$f(r)\frac{p'/c^2(1 - v^2/c^2)}{1 - p'v^2/c^4}|vv_r| \leq f(r)\frac{1 \cdot 1}{1 - a}|vv_r| \leq f(r)\frac{c^2}{2(1 - a)} \quad (\text{by (32)}). \quad (50)$$

Thus, expression (49) becomes

$$\int_0^R f(r)v_t dr + \int_0^R f(r)vv_r dr + \frac{c^2}{2(1 - a)} \int_0^R f(r)dr + \int_0^R f(r)\frac{(1 - v^2/c^2)^2p'}{q(1 - p'v^2/c^4)}\rho_r dr \geq 0. \quad (51)$$

Second, denote the fourth term in expression (51) to be I .

$$I = \int_0^R \frac{f(r)(1-v^2/c^2)^2 \gamma \rho^{\gamma-1}}{(\rho^\gamma/c^2 + \rho)(1-p'v^2/c^4)} d\rho \quad (\text{by (4) and (23)}) \quad (52)$$

$$= \int_0^R \frac{f(r)(1-v^2/c^2)^2}{(1-p'v^2/c^4)} \left(\frac{\gamma \rho^{\gamma-1}}{\rho^\gamma/c^2 + \rho} \right) d\rho \quad (53)$$

$$= \frac{\gamma c^2}{\gamma-1} \int_0^R \frac{f(r)(1-v^2/c^2)^2}{1-p'v^2/c^4} d[\ln(1+\rho^{\gamma-1}/c^2)] \quad (54)$$

$$= -\frac{\gamma c^2}{\gamma-1} \int_0^R \ln(1+\rho^{\gamma-1}/c^2) d\left[\frac{f(r)(1-v^2/c^2)^2}{1-p'v^2/c^4}\right] \quad (\text{by integration by parts}) \quad (55)$$

$$= -\frac{\gamma c^2}{\gamma-1} \int_0^R \ln(1+\rho^{\gamma-1}/c^2) \left[\frac{(1-v^2/c^2)^2}{1-p'v^2/c^4}\right] df(r) \quad (56)$$

$$- \frac{\gamma c^2}{\gamma-1} \int_0^R \ln(1+\rho^{\gamma-1}/c^2) f(r) d\left[\frac{(1-v^2/c^2)^2}{1-p'v^2/c^4}\right] \leq -\frac{\gamma c^2}{\gamma-1} \int_0^R \ln(1+\rho^{\gamma-1}/c^2) f(r) d\left[\frac{(1-v^2/c^2)^2}{1-p'v^2/c^4}\right] \quad (57)$$

$$= -\frac{\gamma c^2}{\gamma-1} \int_0^R \ln(1+\rho^{\gamma-1}/c^2) f(r) \left(\frac{2(1-v^2/c^2)(-(v^2)_r/c^2)}{1-p'v^2/c^4} + (1-v^2/c^2)^2 \frac{d(p'v^2/c^4)}{(1-p'v^2/c^4)^2} \right) dr \quad (58)$$

$$= -\frac{\gamma}{\gamma-1} \int_0^R \ln(1+\rho^{\gamma-1}/c^2) f(r) \left(\frac{2(1-v^2/c^2)(-(v^2)_r)}{1-p'v^2/c^4} + (1-v^2/c^2)^2 \frac{[(p')_r v^2 + (p')(v^2)_r]/c^2}{(1-p'v^2/c^4)^2} \right) dr \quad (59)$$

$$= -\frac{\gamma}{\gamma-1} \int_0^R f(r) \ln(1+\rho^{\gamma-1}/c^2) \frac{(1-v^2/c^2)^2 v^2/c^2}{(1-p'v^2/c^4)^2} (p')_r dr \quad (60)$$

$$- \frac{\gamma}{\gamma-1} \int_0^R f(r) \ln(1+\rho^{\gamma-1}/c^2) \frac{(1-v^2/c^2)(-2+p'v^2/c^4+p'/c^2)}{(1-p'v^2/c^4)^2} (v^2)_r dr.$$

By the elementary inequality

$$\ln(1+x) \leq x, \quad \text{for } x \geq 0, \quad (61)$$

we have

$$\ln(1+\rho^{\gamma-1}/c^2) \leq \rho^{\gamma-1}/c^2 = \frac{p'}{\gamma c^2} < \frac{1}{\gamma}. \quad (62)$$

Thus,

$$I \leq \frac{\gamma}{\gamma-1} \int_0^R f(r) \ln(1 + \rho^{\gamma-1}/c^2) \frac{(1 - v^2/c^2)^2 v^2/c^2}{(1 - p'v^2/c^4)^2} |(p')_r| dr \quad (63)$$

$$+ \frac{\gamma}{\gamma-1} \int_0^R f(r) \ln(1 + \rho^{\gamma-1}/c^2) \frac{(1 - v^2/c^2)(2 + p'v^2/c^4 + p'/c^2)}{(1 - p'v^2/c^4)^2} |(v^2)_r| dr$$

$$\leq \frac{c^2}{(\gamma-1)(1-a)^2} \int_0^R f(r) dr + \frac{4c^2}{(\gamma-1)(1-a)^2} \int_0^R f(r) dr \quad (\text{by (33) and (32)}) \quad (64)$$

$$= \frac{5c^2}{(\gamma-1)(1-a)^2} \int_0^R f(r) dr. \quad (65)$$

It follows that expression (51) becomes

$$\int_0^R f(r) v_t dr + \int_0^R f(r) v v_r dr + C \int_0^R f(r) dr \geq 0, \quad (66)$$

where

$$C := \frac{c^2}{2(1-a)} + \frac{5c^2}{(\gamma-1)(1-a)^2} = \frac{c^2(\gamma + a - \gamma a + 9)}{2(\gamma-1)(1-a)^2} > 0. \quad (67)$$

Third, by integration by parts, we have

$$\int_0^R f(r) v_t dr - \frac{1}{2} \int_0^R v^2 f'(r) dr + C \int_0^R f(r) dr \geq 0. \quad (68)$$

Let

$$H(t) := \int_0^R f(r) v(t, r) dr. \quad (69)$$

Then,

$$H'(t) = \int_0^R f(r) v_t dr, \quad (70)$$

and, by the integral version of the Cauchy inequality, we have

$$H^2(t) = \left(\int_0^R f v dr \right)^2 = \left(\int_0^R \frac{f}{f'} v df \right)^2 \leq \left(\int_0^R \frac{f^2}{f'^2} df \right) \left(\int_0^R v^2 df \right) \quad (71)$$

or

$$H^2(t) \leq \left(\int_0^R \frac{f^2(r)}{f'(r)} dr \right) \left(\int_0^R f'(r) v^2 dr \right). \quad (72)$$

From (70) and (72), (68) becomes

$$H'(t) - \frac{H^2(t)}{2 \int_0^R \frac{f^2(r)}{f'(r)} dr} + C \int_0^R f(r) dr \geq 0 \quad (73)$$

or

$$H'(t) \geq \frac{H^2(t)}{2B_1} - B_2, \quad (74)$$

where

$$B_1 := \int_0^R \frac{f^2(r)}{f'(r)} dr > 0 \quad (75)$$

and

$$B_2 = C \int_0^R f(r) dr > 0. \quad (76)$$

From (74), it is well-known that if

$$H(0) > \sqrt{2B_1B_2}, \quad (77)$$

then the solutions blow up on or before the finite time

$$T := \frac{2B_1H(0)}{H^2(0) - 2B_1B_2} > 0. \quad (78)$$

More precisely, from (77) and the continuity of $H(t)$, $H'(t) > 0$ for all $t \geq 0$. Moreover, from (74),

$$H'(t) \geq \frac{H^2(t)}{2B_1} - B_2 \quad (79)$$

$$= \frac{1}{2B_1} \left(1 - \frac{2B_1B_2}{H^2(0)} \right) H^2(t) + B_2 \left(\frac{H^2(t)}{H^2(0)} - 1 \right) \quad (80)$$

$$\geq \frac{1}{2B_1} \left(1 - \frac{2B_1B_2}{H^2(0)} \right) H^2(t) \quad (81)$$

$$= \frac{H^2(0) - 2B_1B_2}{2B_1H^2(0)} H^2(t). \quad (82)$$

It follows that

$$H(t) \geq \left(\frac{1}{H(0)} - \frac{H^2(0) - 2B_1B_2}{2B_1H^2(0)} t \right)^{-1}. \quad (83)$$

Thus, the blowup time T in (78) is obtained.

The proof is complete. ■

3 Acknowledgement

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